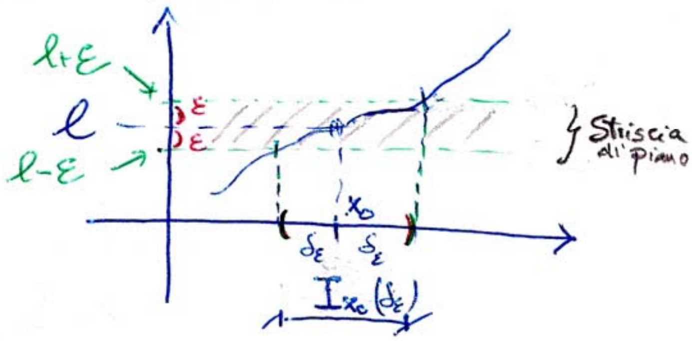
DEFINIZIONE : LIMITE FINITO per $x \to x_0$

$$\lim_{x \to x_0} f(x) = \ell$$



$\forall \varepsilon > 0 \ \exists \delta_\varepsilon > 0$ t.c. $\forall x \in D_f$, $0 < |x - x_0| < \delta_\varepsilon$
$\Rightarrow |f(x) - \ell| < \varepsilon$

$\forall \varepsilon > 0 \Rightarrow$ In qualsiasi modo scelga ε positivo, piccolo a piacere,

$\exists \delta_\varepsilon > 0$ esiste un δ anch'esso positivo, dipendente da ε,

t.c. $\forall x \in D_f$ tale che : per ogni x appartenente

$0 < |x - x_0| < \delta_\varepsilon$

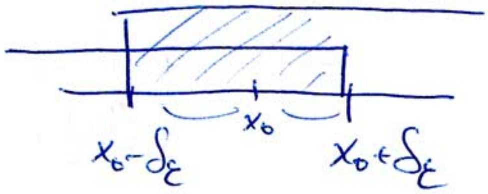
$\Updownarrow$ $\forall x \in I_{x_0}(\delta_\varepsilon)$ $(*)$, $x \neq x_0$ { ad un intorno di x_0 di raggio δ_ε , eventualmente escluso x_0 , si ha :

$|f(x) - \ell| < \varepsilon \iff \ell - \varepsilon < f(x) < \ell + \varepsilon$

ovvero : $f(x)$ sta nella striscia di piano delimitata dalle rette $y = \ell - \varepsilon$ e $y = \ell + \varepsilon$ (v. fig.)

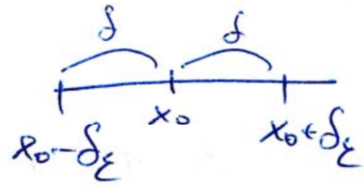
oppure : $f(x) \in I_\ell(\varepsilon)$ intorno di ℓ di raggio ε.

(*) $|x-x_0|<\delta_\varepsilon$ equivale a $\begin{cases} x-x_0<\delta_\varepsilon \\ x-x_0>-\delta_\varepsilon \end{cases}$



$\begin{cases} x<x_0+\delta_\varepsilon \\ x>+x_0-\delta_\varepsilon \end{cases}$

si tratta di un intorno di x_0, di raggio δ_ε



Scrivere $|x-x_0|>0$
vuol dire escludere
il caso in cui $x=x_0$

$|x-x_0|\geq 0$ sempre

$=0$ se e solo se $x=x_0$.

$$0<|x-x_0|<\delta_\varepsilon \iff x\in I_{x_0}(d_\varepsilon)$$
$$x\neq x_0.$$

Allo stesso modo la
scrittura $|f(x)-\ell|<\varepsilon$

equivale a: $-\varepsilon<f(x)-\ell<\varepsilon$

$$\Rightarrow \quad \ell-\varepsilon<f(x)<\ell+\varepsilon$$
$$\Rightarrow f(x)\in I_\ell(\varepsilon)$$

Verificare, tramite def, che: $\lim_{x \to 3} (5x - 1) = 14$

Devo dim. che

$$\forall \varepsilon > 0 \ \exists \delta_\varepsilon > 0 \ : \ \text{per} \ \overbrace{0 < |x-3| < \delta_\varepsilon}^{I_3(\delta_\varepsilon),\ x \neq 3}$$

$$\Downarrow$$

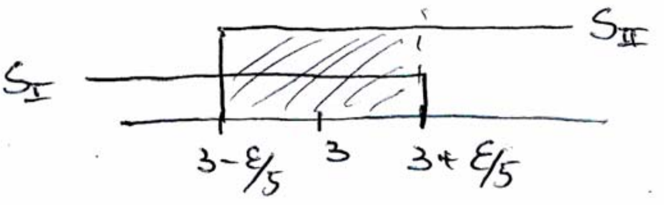
$$|5x - 1 - 14| < \varepsilon$$

$$|5x - 15| < \varepsilon$$

è equivalente al seguente sistema:

$$\begin{cases} 5x - 15 < \varepsilon \\ 5x - 15 > -\varepsilon \end{cases} \Rightarrow \begin{cases} 5x < 15 + \varepsilon \\ 5x > 15 - \varepsilon \end{cases}$$

$$\begin{cases} x < \dfrac{15+\varepsilon}{5} \\ x > \dfrac{15-\varepsilon}{5} \end{cases} \Rightarrow \begin{cases} x < 3 + \dfrac{\varepsilon}{5} \\ x > 3 - \dfrac{\varepsilon}{5} \end{cases}$$



$$\Downarrow$$

$$\underbrace{3 - \frac{\varepsilon}{5} < x < 3 + \frac{\varepsilon}{5}}$$

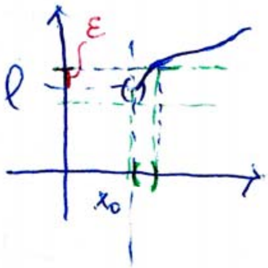
ovvero in un intorno di 3, di raggio $\varepsilon/5$

$\Rightarrow$ Basta prendere $\delta_\varepsilon = \varepsilon/5$

Abbiamo così verificato che, in qualsiasi modo si scelga un numero $\varepsilon > 0$, piccolo a piacere, la disequaz. $|5x - 15| < \varepsilon$ risulta soddisfatta per tutti i valori di x (escluso al più 3) che formano un intorno completo del punto 3.

LIMITE DESTRO :

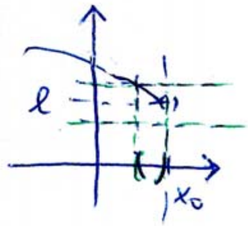
$$\lim_{x \to x_0^+} f(x) = \ell$$



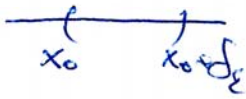
La funz. sta nella striscia $(\ell-\varepsilon ; \ell+\varepsilon)$ in un intorno destro di x_0.

$$\forall \varepsilon > 0 \ \exists \delta_\varepsilon > 0 : \forall x \in D_f, \ \overbrace{x_0 < x < x_0 + \delta_\varepsilon}^{I^+_{x_0}(\delta_\varepsilon)}$$

$$\Rightarrow |f(x) - \ell| < \varepsilon$$



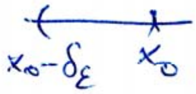
intorno dx



LIMITE SINISTRO :

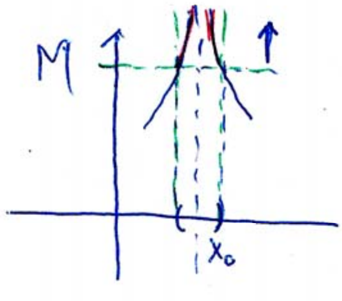
$$\lim_{x \to x_0^-} f(x) = \ell$$

intorno sin



$$\forall \varepsilon > 0 \ \exists \delta_\varepsilon > 0 : \forall x \in D_f, \ x_0 - \delta_\varepsilon < x < x_0 \Rightarrow |f(x) - \ell| < \varepsilon$$

LIMITE INFINITO per $x \to x_0$



Asint. Verticale $\Leftarrow$ $\lim_{x \to x_0} f(x) = +\infty$

$\forall M > 0 \ \exists \delta_M > 0$ t.c. $\forall x \in D_f$ con

$0 < |x - x_0| < \delta_M$

$\Rightarrow f(x) > M$

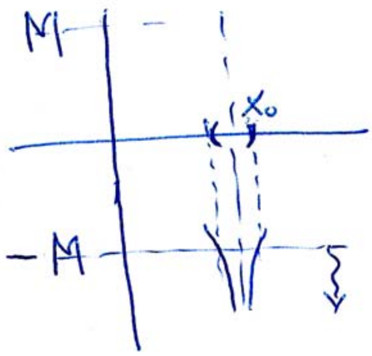
$\forall M > 0$ sufficientemente grande esiste un intorno di x_0, di raggio δ_M, (con $x \neq x_0$) per ogni x del quale risulta: $f(x) > M$.

$$\lim_{x \to x_0} f(x) = -\infty$$

$\forall M > 0 \ \exists \delta_M > 0$

t.c. $\forall x \in D_f, \ 0 < |x - x_0| < \delta_M$

$\Rightarrow f(x) < -M$



nell'intorno di x_0

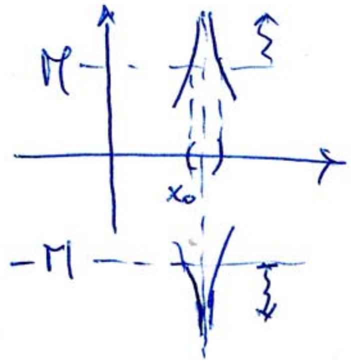
$f(x) < -M$

$$\lim_{x \to x_0} f(x) = \infty$$

$\forall M > 0 \ \exists \delta_M > 0$

t.c. $\forall x \in D_f, \ 0 < |x - x_0| < \delta_M$

$\Rightarrow |f(x)| > M$



nell'intorno di x_0

$f(x) > M$

$\cup$ o $f(x) < -M$

$\Downarrow$

$|f(x)| > M$

$$\lim_{x \to x_0^+} f(x) = \infty \quad \longrightarrow \text{ si tratta di un limite dx} \Rightarrow x_0 < x < x_0 + \delta_M$$

$\Rightarrow$ la disuguaglianza sarà verificata in $I^+_{x_0}(\delta_M)$ $\Updownarrow$

$$\lim_{x \to x_0^-} f(x) = \infty$$

$\hookrightarrow$ limite sin. $\Rightarrow$ La disuguaglianza dovrà essere verificata in un intorno sin $\Rightarrow x_0 - \delta_M < x < x_0$

$I^-_{x_0}(\delta_M)$

Comp. 16/7/13

Es/ Verificare, tramite def, che $\lim_{x\to 2^+} \frac{1}{2-x} = -\infty$

Devo dim. che

$$\forall M>0 \ \exists\, \delta_M > 0 \text{ t.c. per } \overbrace{2 < x < 2+\delta_M}^{I_+(2),\ x\neq 2} \Rightarrow \frac{1}{2-x} < -M$$

$$\frac{1}{2-x} < -M$$

$$\frac{1}{2-x} + M < 0 \quad \Rightarrow \quad \frac{1+2M-Mx}{2-x} < 0$$

Diseq. fratta $\Rightarrow$ Studio il segno del Num. e del Denom.

$N > 0 \Rightarrow 1+2M-Mx > 0$

$Mx < 1+2M$

$\begin{pmatrix} M>0 \\ \text{per ip} \end{pmatrix} \quad x < \frac{1+2M}{M} = 2+\frac{1}{M}$

$D > 0 \Rightarrow 2-x > 0 \Rightarrow x < 2$

		2		$2+\frac{1}{M}$	
	$+$		$\bigcirc\!\!\!-$		$+$
D	$+$		$-$		$-$
N	$+$		$+$		$-$

$\Rightarrow \quad 2 < x < 2+\frac{1}{M}$

$\therefore$ Basta prendere $\delta_M = \frac{1}{M}$.

Esercizio (da fare):

Verificare che $\lim_{x\to -1^-} \frac{2-x}{x+1} = -\infty$

$$\lim_{x \to +\infty} f(x) = \ell$$

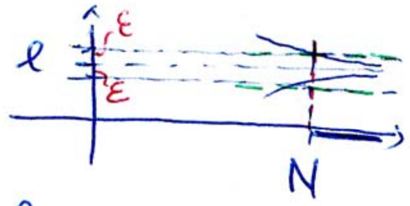
LIMITE FINITO
per $x \to \infty$

Asint. Orizzontale

$\forall \varepsilon > 0 \quad \exists N > 0$ t.c.

$\forall x \in D_f \,,\ x > N$

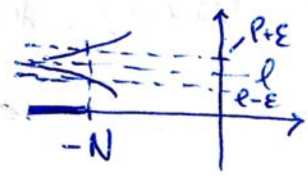
$\Rightarrow \quad |f(x) - \ell| < \varepsilon$



La funz. è compresa nella striscia $(\ell - \varepsilon ; \ell + \varepsilon)$ per $x > N$

numero positivo sufficientemente grande

$$\lim_{x \to -\infty} f(x) = \ell$$

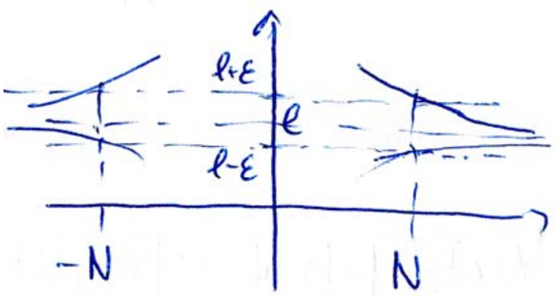


$\forall \varepsilon > 0 \quad \exists N > 0$ t.c.

$\forall x \in D_f \,,\ x < -N \Rightarrow |f(x) - \ell| < \varepsilon$

$$\lim_{x \to \infty} f(x) = \ell$$

$\forall \varepsilon > 0 \quad \exists N > 0$ t.c. $\forall x \in D_f \,,\ \underbrace{|x| > N} \Rightarrow |f(x) - \ell| < \varepsilon$



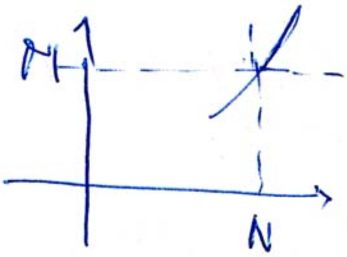
la funz. sta nella striscia $(\ell - \varepsilon ; \ell + \varepsilon)$

per $x > N \ \cup \ x < -N$

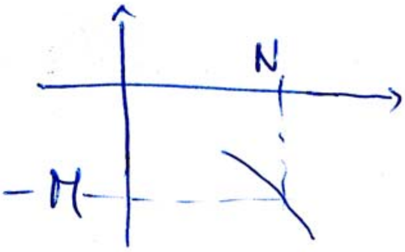
$\Rightarrow$ per $|x| > N$

LIMITE INFINITO per $x \to \infty$

$$\lim_{x \to +\infty} f(x) = +\infty$$



$\forall M > 0 \ \exists N > 0$ t.c. $\forall x \in D_f,\ x > N$
$\Rightarrow f(x) > M$

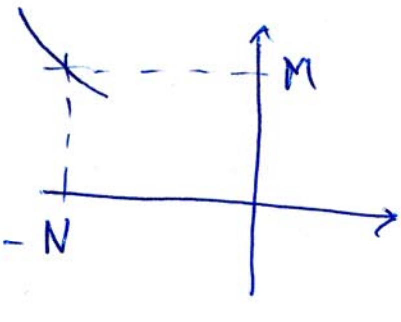


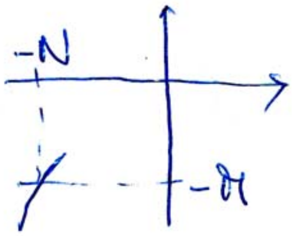
$$\lim_{x \to +\infty} f(x) = -\infty$$

$\forall M > 0 \ \exists N > 0$ t.c. $\forall x \in D_f,\ x > N \Rightarrow f(x) < -M$

$$\lim_{x \to -\infty} f(x) = +\infty$$

$\forall M > 0 \ \exists N > 0$ t.c. $\forall x \in D_f,\ x < -N$
$\Rightarrow f(x) > M$





$$\lim_{x \to -\infty} f(x) = -\infty$$

$\forall M > 0 \ \exists N > 0$ t.c. $\forall x \in D_f,\ x < -N$
$\Rightarrow f(x) < -M$

In generale:

$\lim_{x \to \infty} f(x) = \infty$ se $\forall \ > 0 \ \exists N > 0$ t.c. $\forall x \in D_f,\ |x| > N \Rightarrow |f(x)| > M$

Es Verificare che: $\displaystyle\lim_{x \to -\infty} \frac{x^2}{x-1} = -\infty$

Dobbiamo dim. che:

$\forall M > 0 \ \exists N > 0$ t.c. per $x < -N \Rightarrow \dfrac{x^2}{x-1} < -M$

(*) $\dfrac{x^2}{x-1} < -M$

$$\frac{x^2}{x-1} + M < 0 \Rightarrow \frac{x^2 + Mx - M}{x-1} < 0$$

Studio del segno:

Num $> 0 \Rightarrow x^2 + Mx - M > 0$

$\Delta = M^2 + 4M$ (> 0 perché $M > 0$)

$x_{1,2} = \dfrac{-M \pm \sqrt{M^2+4M}}{2} \Rightarrow \dfrac{-M + \overbrace{\sqrt{M^2+4M}}^{(>M)}}{2}$ (>0)

$\dfrac{-M - \sqrt{M^2+4M}}{2}$

Num $> 0 \ \forall$ per i valori esterni.

Den $> 0 \Rightarrow x > 1$

	(−)	+	(−)	+
D	−	−	−	+
N	+	−	+	+

Punti: $-\dfrac{M+\sqrt{M^2+4M}}{2}$, 0, $\dfrac{-M+\sqrt{M^2+4M}}{2}$, 1

Ponendo $N = \dfrac{M + \sqrt{M^2+4M}}{2}$ la disequazione (*) è soddisfatta per $x < -N \Rightarrow$ il limite è verificato

Osservaz. $\dfrac{-M + \sqrt{M^2+4M}}{2} \overset{?}{<} 1$ sì

$-M + \sqrt{M^2+4M} < 2 \Rightarrow \sqrt{M^2+4M} < 2 + M$

Dis. irrazionale equivalente a $\begin{cases} M^2+4M \geq 0 \\ 2+M > 0 \\ M^2+4M < (2+M)^2 \end{cases}$ ($M > 0$)

$$\begin{cases} M(M+4) \geq 0 & \forall M \ (>0) \\ 2+M > 0 & \forall M \ (>0) \\ \cancel{M^2}+\cancel{4M} < 4 + \cancel{M^2}+\cancel{4M} \Rightarrow 0 < 4 & \forall M \end{cases}$$

$$\lim_{x\to+\infty} \frac{9x^3-7x+5}{-2x^3+4x^2-1} = \lim_{x\to+\infty} \frac{\cancel{x^3}\left(9-\frac{7}{x^2}+\frac{5}{x^3}\right)}{\cancel{x^3}\left(-2+\frac{4}{x}-\frac{1}{x^3}\right)} = -\frac{9}{2}$$

$$\lim_{x\to 0} \frac{\sqrt{1+x+x^2}-1}{x} \overset{(\text{"razion."})}{=} \lim_{x\to 0} \frac{\left(\sqrt{1+x+x^2}-1\right)\left(\sqrt{1+x+x^2}+1\right)}{x\left(\sqrt{1+x+x^2}+1\right)} =$$

$$= \lim_{x\to 0} \frac{\cancel{1}+x+x^2-\cancel{1}}{x\left(\sqrt{1+x+x^2}+1\right)} = \lim_{x\to 0} \frac{\cancel{x}(1+x)}{\cancel{x}\left(\sqrt{1+x+x^2}+1\right)} = \frac{1}{2}$$

$$\lim_{x\to 1} \frac{x^3-3x+2}{x^3-x^2-x+1} = \left(\frac{0}{0}\right)$$

Vanno scomposti in fattori sia il Num. che il Denom.

Num $x^3-3x+2 \longrightarrow$ Ruffini

Si cercano gli zeri del polinomio fra i divisori del termine noto: $\pm 1, \pm 2$

$P(1) = 1-3+2 = 0$ ok!

$\Rightarrow$ Il polinomio è divisibile per $(x-1)$

Il quoziente si trova con la tab. di Ruffini.

$$\begin{array}{c|ccc|c} & 1 & 0 & -3 & 2 \\ 1 & & 1 & 1 & -2 \\ \hline & 1 & 1 & -2 & 0 \end{array}$$

deve tornare zero.

$\underbrace{1 \quad 1 \quad -2}_{\text{coeff. del quoziente}}$

$\left.\begin{array}{l} s=1 \\ p=-2 \end{array}\right\} (+2);(-1)$

$$\Rightarrow x^3-3x+2 = (x-1)\cdot(x^2+x-2) = (x-1)^2(x+2)$$

Dim

$$x^3 - x^2 - x + 1 = x^2(x-1) - (x-1) =$$

Raccoglim. successivo

$$= (x-1)(x^2-1) =$$

$$= (x-1)(x+1)(x-1)$$

Somma per differenza o diff. di quad.

$$= (x-1)^2(x+1)$$

$$\lim_{x\to 1} \frac{x^3-3x+2}{x^3-x^2-x+1} = \lim_{x\to 1} \frac{\cancel{(x-1)^2}(x+2)}{\cancel{(x-1)^2}(x+1)} = \frac{3}{2}$$

$$\lim_{x\to 0} \frac{1-\cos^3 x}{3x^2} = \lim_{x\to 0} \frac{(1-\cos x)(1+\cos x+\cos^2 x)}{3x^2} =$$

$$= \lim_{x\to 0} \underbrace{\frac{1-\cos x}{x^2}}_{\downarrow\ 1/2} \cdot \frac{1+\cos x+\cos^2 x}{3} = \frac{1}{2}\cdot\frac{\cancel{3}}{\cancel{3}} = \frac{1}{2}$$

$$\lim_{x\to 0} \frac{\sin^2 2x}{1-\cos 3x} =$$

$$\boxed{\begin{aligned}&\lim_{x\to 0}\frac{1-\cos x}{x^2} = \frac{1}{2}\\ &\lim_{x\to 0}\frac{x^2}{1-\cos x} = 2\end{aligned}}$$

$$= \lim_{x\to 0} \frac{\sin 2x}{2x}\cdot 2x \cdot \frac{\sin 2x}{2x}\cdot 2x \cdot \frac{9}{1-\cos 3x}\cdot\frac{1}{9} =$$

$$\downarrow \qquad\qquad \swarrow(*)$$

$$= 1\cdot 2\cdot 1\cdot 2\cdot 2\cdot \frac{1}{9} = \frac{8}{9}$$

$$\lim_{x\to 0} \frac{\operatorname{sen} 2x}{\operatorname{tg} x} = \lim_{x\to 0} \overset{\overset{1}{\uparrow}}{\frac{\operatorname{sen} 2x}{2x}} \cdot 2\overset{\overset{1}{\uparrow}}{x \frac{\cos x}{\operatorname{sen} x}} = 2$$

$$\lim_{x\to 0} \frac{1-\cos x}{\operatorname{sen} 4x} = \lim_{x\to 0} \overset{\overset{0}{\uparrow}}{\frac{1-\cos x}{x}} \cdot \frac{x}{4} \cdot \overset{\overset{1}{\uparrow}}{\frac{4}{\operatorname{sen} 4x}} =$$

$$= 0 \cdot \frac{1}{4} \cdot 1 = 0.$$

$$\lim_{x\to 0} \frac{1-\cos x}{x^2} = \lim_{x\to 0} \frac{(1-\cos x)(1+\cos x)}{x^2(1+\cos x)} =$$

$$= \lim_{x\to 0} \frac{1-\cos^2 x}{x^2(1+\cos x)} = \lim_{x\to 0} \underset{\underset{1}{\downarrow}}{\frac{\operatorname{sen}^2 x}{x^2}} \frac{1}{(1+\cos x)} = \frac{1}{2}$$

oppure: sfruttando la formula di bisezione $\operatorname{sen}^2 \alpha = \frac{1-\cos 2\alpha}{2}$

$$\Rightarrow \frac{1-\cos x}{2} = \operatorname{sen}^2 \frac{\alpha}{2}$$

$$\Rightarrow \lim_{x\to 0} \frac{1-\cos x}{x^2} = \lim_{x\to 0} \frac{2\operatorname{sen}^2(x/2)}{x^2} = \lim_{x\to 0} \frac{\operatorname{sen}^2\left(\frac{x}{2}\right)}{\frac{x^2}{2}}$$

$$= \lim_{x\to 0} \left[\frac{\operatorname{sen}\frac{x}{2}}{\frac{x}{2}} \cdot \frac{\operatorname{sen}\frac{x}{2}}{\frac{x}{2}\cdot 2} \right] = \frac{1}{2}$$

$$\lim_{x\to 0} \frac{x^2 \ln(1+2x)}{[2\cos(3x)-2]\operatorname{sen} x} =$$

$$= \lim_{x\to 0} \frac{\overbrace{\frac{x}{\operatorname{sen} x}}^{\downarrow 1} \cdot x \ln(1+2x)}{2[\cos 3x - 1]} =$$

$$= \lim_{x\to 0} \left(- \frac{x \ln(1+2x)}{2\,(1-\cos 3x)} \cdot \frac{9x}{9x} \right)$$

$$= \lim_{x\to 0} \left(- \underbrace{\frac{\ln(1+2x)}{2x}}_{\downarrow 1} \cdot \underbrace{\frac{9x^2}{1-\cos 3x}}_{\downarrow 2} \cdot \frac{1}{9} \right) = -\frac{2}{9}$$

$$\lim_{x\to 0} \frac{3x + \operatorname{tg} x}{\operatorname{sen} x + \operatorname{tg}^2 x} = \lim_{x\to 0} \frac{x\left(3 + \overbrace{\frac{\operatorname{tg} x}{x}}^{1\uparrow}\right)}{\operatorname{sen} x + \frac{\operatorname{sen}^2 x}{\cos^2 x}} =$$

1, Infatti: $\lim_{x\to 0} \overbrace{\frac{\operatorname{sen} x}{x \cos x}}^{1\uparrow} = 1$

$$= \lim_{x\to 0} \frac{\overbrace{\frac{x}{\operatorname{sen} x}}^{1\uparrow} \left(3 + \overbrace{\frac{\operatorname{tg} x}{x}}^{1\uparrow}\right)}{\left(1 + \frac{\operatorname{sen} x}{\cos^2 x}\right)} = \frac{1\cdot(3+1)}{\left(1 + \frac{0}{1}\right)} = 4$$