Es/ Verificare che: $\lim_{x\to 3} \frac{1}{x} = \frac{1}{3}$

Devo dim. che $\forall \varepsilon > 0 \ \exists \delta_\varepsilon > 0 : \forall x \neq 0, \underbrace{0 < |x-3| < \delta_\varepsilon}_{\substack{I_3(\delta_\varepsilon) \\ x \neq 3}}$

si ha: $\left| \frac{1}{x} - \frac{1}{3} \right| < \varepsilon$

$$\left| \frac{1}{x} - \frac{1}{3} \right| < \varepsilon$$

$$-\varepsilon < \frac{1}{x} - \frac{1}{3} < \varepsilon$$

$$\frac{1}{3} - \varepsilon < \frac{1}{x} < \frac{1}{3} + \varepsilon$$

(non è restrittivo considerare $0 < \varepsilon < \frac{1}{3}$)

$$\frac{1-3\varepsilon}{3} < \frac{1}{x} < \frac{1+3\varepsilon}{3} \Rightarrow \frac{3}{1+3\varepsilon} < x < \frac{3}{1-3\varepsilon}$$

Osservazione:

$$\frac{3}{1+3\varepsilon} = \frac{3+9\varepsilon-9\varepsilon}{1+3\varepsilon} = \frac{3(1+3\varepsilon)-9\varepsilon}{1+3\varepsilon} = 3 - \frac{9\varepsilon}{1+3\varepsilon}$$

$$\frac{3}{\underbrace{1-3\varepsilon}_{>0 \Rightarrow 3\varepsilon < 1,\ \varepsilon < 1/3 \text{ ok}}} = \frac{3-9\varepsilon+9\varepsilon}{1-3\varepsilon} = \frac{3(1-3\varepsilon)+9\varepsilon}{1-3\varepsilon} = 3 + \frac{9\varepsilon}{1-3\varepsilon}$$

La disequaz. $\left| \frac{1}{x} - \frac{1}{3} \right| < \varepsilon$ è soddisfatta per tutti i valori di x che formano un intorno completo del pt 3.

Es. Verificare che $\lim_{x \to 0} 2^x = 1$

Devo verificare che $\forall \varepsilon > 0 \ \exists \delta_\varepsilon > 0 : \forall x \in \mathbb{R}, \underbrace{0 < |x| < \delta_\varepsilon}_{\text{ovvero in un } I_0(\delta_\varepsilon),\ x \neq 0}$
si ha: $|2^x - 1| < \varepsilon$

$$|2^x - 1| < \varepsilon$$

$$\begin{cases} 2^x - 1 < \varepsilon & \Rightarrow \ 2^x < 1 + \varepsilon \\ 2^x - 1 > -\varepsilon & \Rightarrow \ 2^x > 1 - \varepsilon \end{cases}$$

$$1 - \varepsilon < 2^x < 1 + \varepsilon$$

Passando al $\log_2$

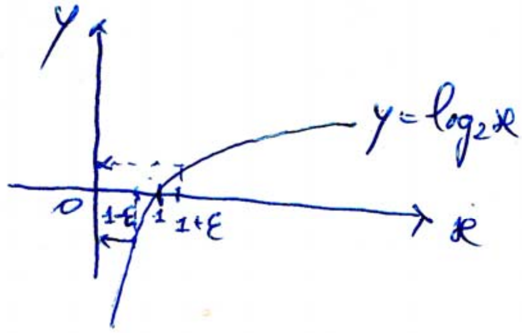
$$\log_2(1-\varepsilon) < \log_2 2^x < \log_2(1+\varepsilon)$$

$$\log_2(1-\varepsilon) < x < \log_2(1+\varepsilon)$$

$\downarrow$

$0 < \varepsilon < 1$

che formano un intorno completo del punto 0.



Es/ Verificare che: $\lim_{x \to +\infty} \left(\sqrt{x+1} - \sqrt{x}\right) = 0$

$\swarrow$

Dobbiamo verificare che $\forall \varepsilon > 0 \quad \exists N > 0$ t.c.

$$\forall x \in D_f, \quad x > N \Rightarrow \left|\sqrt{x+1} - \sqrt{x}\right| < \varepsilon$$

Oss.

$$\left|\sqrt{x+1} - \sqrt{x}\right| = \left|\frac{\left(\sqrt{x+1} - \sqrt{x}\right)\left(\sqrt{x+1} + \sqrt{x}\right)}{\sqrt{x+1} + \sqrt{x}}\right| =$$

$$= \left|\frac{\cancel{x}+1 - \cancel{x}}{\sqrt{x+1} + \sqrt{x}}\right| = \left|\underbrace{\frac{1}{\sqrt{x+1} + \sqrt{x}}}_{>0}\right| = \frac{1}{\sqrt{x+1} + \sqrt{x}} \leq$$

$$\underbrace{\leq}_{\text{(maggiorazione)}} \frac{1}{\sqrt{x} + \sqrt{x}} = \frac{1}{2\sqrt{x}}$$

$$\Rightarrow \left|\sqrt{x+1} - \sqrt{x}\right| \leq \underbrace{\frac{1}{2\sqrt{x}} < \varepsilon}$$

$\swarrow$

$$2\sqrt{x} > \frac{1}{\varepsilon}$$

$$4x > \frac{1}{\varepsilon^2} \Rightarrow x > \frac{1}{4\varepsilon^2} \Rightarrow \text{Basta prendere } N = \frac{1}{4\varepsilon^2}.$$

Per definizione

Data $f(x)$ definita in un intorno di $x_0 = I(x_0)$

e supposto che esista

$f^{(k)}(x_0)$, $k = 0, 1, \dots n$ $(n \in N)$

$$\sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!} \cdot h^k$$

si chiama

POLINOMIO DI TAYLOR di ordine n della f relativo al punto x_0.

TEOREMA = FORMULA di TAYLOR CON IL TERMINE COMPLEMENTARE di PEANO

(con errore nella forma di Peano).

Hp: Sia $f(x)$ definita in un intorno di $x_0 = I(x_0)$

$\exists\, f^{(k)}(x)$ $\forall x \in I(x_0)$, $k = n-1$

$\exists\, f^{n}(x_0)$

Th: $f(x_0+h) = P_n^f(x_0, h) + R_n(h) =$

$$= f(x_0) + h f'(x_0) + \frac{h^2}{2!} f''(x_0) + \dots + \frac{h^n}{n!} f^{(n)}(x_0) + R_n(h)$$

dove $R_n(h)$ è un infinitesimo di ordine superiore a h^n, ovvero:

$$\lim_{x \to x_0} \frac{R_n(h)}{h^n} = 0$$

Ponendo $x_0 = 0$ e $h = x$ si ha la FORMULA di MAC-LAURIN

$$f(x) = \sum_{k=0}^{n} \frac{f^{(k)}(0)}{k!} \cdot x^k + R_n(x) \quad , \text{ dove } \lim_{x \to 0} \frac{R_n(x)}{x^n} = 0$$

Indicando $x = x_0 + h$ si può scrivere: $f(x) = \sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!} \cdot (x - x_0)^k + R_n(x - x_0)$

TAYLOR

oppure : $R_n(h)$

$$f(x_0+h) = P_n^{f}(x_0, h) + o(h^n)$$

"o piccolo"

$$\boxed{\lim \frac{o(h^n)}{h^n} = 0} \Longleftarrow$$ è un infinitesimo di grado superiore a h^n

Def:

α è un infinitesimo se $\lim \alpha = 0$

Se α, β sono infinitesimi

α è un infinitesimo di grado sup. a β se $\lim \frac{\alpha}{\beta} = 0$

Se $\lim \frac{\alpha}{\beta} = \ell \neq 0 \Rightarrow \alpha, \beta$ sono infinitesimi dello stesso ordine

Se $\lim \frac{\alpha}{\beta} = \infty$

$\searrow$ α è un infinitesimo di ordine inf. a β

Es. $\ln(1+x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \dots$

Supponiamo di arrestare lo sviluppo al II termine :

$$\ln(1+x) = x - \frac{1}{2}x^2 + R_2(x)$$

$$R_2(x) = o(x^2)$$

$$\ln(1+x) \simeq x - \frac{1}{2}x^2$$

La parte trascurata è un infinitesimo di ordine superiore al II.

$$\lim_{x\to 0} \frac{R_2(x)}{x^2} = 0 \Longleftarrow$$

Infatti :

$$\lim_{x\to 0} \frac{\frac{1}{3}x^3 - \frac{1}{4}x^4 + \dots}{x^2} =$$

$$= \lim_{x\to 0} \frac{\frac{1}{3}\cancel{x^3}}{\cancel{x^2}} = 0 .$$

$$f(x_0+h) = \underbrace{P_n^f(x_0, h)}_{\sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!} h^k} + o(h^n)$$

$1! = 1$

$2! = 2 \cdot 1 = 2$

$3! = 3 \cdot 2 = 6$

$0! \underset{def}{=} 1$

"!" = "fattoriale"

$$k! = k \cdot (k-1)(k-2) \cdot \ldots \cdot 2 \cdot 1$$

$$4! = 4 \cdot 3 \cdot 2 \cdot 1 = 24$$

derivata di ordine k

$$\sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!} \cdot h^k =$$

"Sommatoria"

$$= f(x_0) + f'(x_0)\, h + f''(x_0) \cdot \frac{h^2}{2!} + \dots$$

$$+ \dots + f^{(n)}(x_0) \frac{h^n}{n!}$$

def

TAYLOR

$$f(x_0+h) = f(x_0) + h\, f'(x_0) + \frac{h^2}{2!} f''(x_0) + \dots$$

$$+ \dots + \frac{h^n}{n!} f^{(n)}(x_0) + o(h^n)$$

$$f(x_0+h) = \sum_{k=0}^{n} \frac{f^{(k)}(x_0)\, h^k}{k!} + o(h^n)$$

Ponendo $x = x_0 + h \Rightarrow h = x - x_0$

$$f(x) = f(x_0) + f'(x_0) \cdot (x - x_0) + f''(x_0) \cdot \frac{(x-x_0)^2}{2!} + \dots$$

$$+ \dots + f^{(n)}(x_0) \cdot \frac{(x-x_0)^n}{n!} + o\left((x-x_0)^n\right)$$

$$f(x) = \sum_{k=0}^{n} \frac{f^{(k)}(x_0) \cdot (x-x_0)^k}{k!} + R_n(x-x_0)$$

Ponendo $\left.\begin{array}{l} x_0 = 0 \\ h = x \end{array}\right\} \; x_0 + h = x$

Si ottiene la formula di MAC-LAURIN

$$\boxed{f(x) = f(0) + f'(0)\cdot x + f''(0)\cdot\frac{x^2}{2!} + f'''(0)\frac{x^3}{3!} + \dots + f^{(n)}(0)\cdot\frac{x^n}{n!} + o(x^n)}$$

Esempio:

$$\cos x = 1 - \frac{x^2}{2} + R_2(x)$$

$$1 - \cos x^5 = ?$$

$$o\left((x^5)^2\right) = o\left(x^{10}\right) = R_{10}(x)$$

$$\cos x^5 = 1 - \frac{(x^5)^2}{2} + R_{10}(x)$$

$$1 - \cos x^5 = 1 - \left(1 - \frac{x^{10}}{2}\right) = \frac{x^{10}}{2} + R_{10}(x)$$

$$R_{10}(x) = ? \Rightarrow \lim_{x \to 0} \frac{R_{10}(x)}{x^{10}} = 0$$

infinitesimo di ordine sup. a x^{10}.

Es: calcolo di un limite utilizzando gli sviluppi di Taylor:

$$\lim_{x\to 0}\left(\frac{\operatorname{sen} 2x}{2x}\right)^{\frac{1}{x^2}} = 1^{\infty} \quad \text{f. ind.}$$

limite notevole: $\lim_{x\to 0}\frac{\operatorname{sen} x}{x} = 1$

$$= \lim_{x\to 0} e^{\frac{1}{x^2}\ln\left(\frac{\operatorname{sen} 2x}{2x}\right)} =$$

$$= e^{\lim_{x\to 0}\frac{1}{x^2}\cdot\ln\left(\frac{\operatorname{sen} 2x}{2x}\right)}$$

$$\operatorname{sen} x = x - \frac{1}{6}x^3 + \overset{o(x^3)}{R_3(x)} \quad \text{(Tab.)}$$

$$\operatorname{sen} 2x = 2x - \frac{1}{6}(2x)^3 + R_3(x) \qquad k\,o(x^3) = o(x^3)$$

$$x^{\alpha}\cdot o(x^{\beta}) = o(x^{\alpha+\beta})$$

$$\frac{\operatorname{sen} 2x}{2x} = \frac{2x - \frac{4}{3}x^3 + R_3(x)}{2x} = 1 - \frac{2}{3}x^2 + R_2(x)$$

$$\frac{o(x^3)}{x} = o(x^2)$$

$$\Rightarrow \lim_{x\to 0}\frac{\ln\left(\frac{\operatorname{sen} 2x}{2x}\right)}{x^2} = \lim_{x\to 0}\frac{\ln\left(1 - \frac{2}{3}x^2\right)}{x^2} =$$

$$\ln(1+x) = x + \underset{o(x)}{\underset{\shortparallel}{R_1(x)}}$$

v. Tab.

$$\boxed{\ln(1+x) = x - \frac{1}{2}x^2 + \dots}$$

$$\ln\left(1-\frac{2}{3}x^2\right) = -\frac{2}{3}x^2 + R_2(x)$$

L'eventuale termine successivo sarebbe di quarto grado, quindi è trascurabile.

(*)

$$= \lim_{x\to 0} \frac{-\frac{2}{3}x^2 + R_2(x)}{x^2} =$$

(*) Oppure x i lim. notevoli:

$$= \lim_{x\to 0} \frac{\ln\left(1-\frac{2}{3}x^2\right)}{-\frac{2}{3}x^2}\cdot\left(-\frac{2}{3}\right) = -\frac{2}{3}$$

$$= \lim_{x\to 0} \frac{-\frac{2}{3}\cancel{x^2}}{\cancel{x^2}} + \underbrace{\lim_{x\to 0} \frac{R_2(x)}{x^2}}_{\downarrow\ 0} = -\frac{2}{3}$$

$$\Rightarrow \lim_{x\to 0} \left(\frac{\operatorname{sen} 2x}{2x}\right)^{\frac{1}{x^2}} = e^{-2/3} =$$

$$= \frac{1}{e^{2/3}} = \frac{1}{\sqrt[3]{e^2}}$$

Es Trovare lo sviluppo di Taylor in $x_0 = 0$ fino al III ordine della funzione $f(x) = (1+x)^\alpha$

$$\boxed{f(x) = f(0) + f'(0)\cdot x + f''(0)\cdot \frac{x^2}{2} + f'''(0)\cdot \frac{x^3}{3!} + R_3(x)}$$

$f(0) = 1$

$f'(x) = \alpha(1+x)^{\alpha-1} \Rightarrow f'(0) = \alpha$

$f''(x) = \alpha(\alpha-1)(1+x)^{\alpha-2} \Rightarrow f''(0) = \alpha\cdot(\alpha-1)$

$f'''(x) = \alpha(\alpha-1)(\alpha-2)(1+x)^{\alpha-3} \Rightarrow f'''(0) = \alpha(\alpha-1)(\alpha-2)$

$$\Rightarrow (1+x)^\alpha = 1 + \alpha x + \frac{\alpha(\alpha-1)}{2}x^2 + \frac{\alpha(\alpha-1)(\alpha-2)}{6}x^3 + R_3(x)$$

Sfruttando lo svil. di Taylor appena ricavato, arrestato la II ordine, calcolare una approssimazione ($\in \mathbb{Q}$) di $\frac{1}{\sqrt[3]{1{,}1}}$.

$$\frac{1}{\sqrt[3]{1{,}1}} = \frac{1}{\sqrt[3]{1+0{,}1}} = (1+0{,}1)^{-1/3} \Rightarrow$$

Si può ricavare una sua approssimazione dallo sviluppo di Taylor di $(1+x)^\alpha$ ponendo $\alpha = -1/3$ e $x = 1/10$.

$$\Rightarrow \frac{1}{\sqrt[3]{1{,}1}} \simeq 1 - \frac{1}{3}\cdot\frac{1}{10} + \frac{-\frac{1}{3}\cdot\left(-\frac{1}{3}-1\right)}{2}\cdot\left(\frac{1}{10}\right)^2 =$$

$$= 1 - \frac{1}{30} + \frac{2}{900} = \frac{872}{900} = \frac{218}{225}$$

Infatti: $\frac{218}{225} = 0{,}96\bar{8} \simeq 1/\sqrt[3]{1{,}1} = 0{,}9687293\ldots$ $(Er < 10^{-3})$

Sviluppi di funzioni mediante la formula di Taylor (per $x \to 0$)

A) <u>Funzioni elementari</u> :

1) $e^x = 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{24}x^4 + \cdots\cdots + \frac{1}{n!}x^n + \cdots\cdots$

2) $\log(1 + x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \cdots\cdots + \frac{(-1)^{n+1}}{n}x^n + \cdots\cdots$

2') $\log(1 - x) = -x - \frac{1}{2}x^2 - \frac{1}{3}x^3 - \frac{1}{4}x^4 - \cdots\cdots - \frac{1}{n}x^n - \cdots\cdots$

2'') $\log\left(\frac{1 + x}{1 - x}\right) = 2x + \frac{2}{3}x^3 + \frac{2}{5}x^5 + \cdots\cdots + \frac{2}{(2n + 1)}x^{2n+1} + \cdots\cdots$

3) $(1 + x)^\alpha = 1 + \underset{=\binom{\alpha}{1}}{\alpha x} + \underset{=\binom{\alpha}{2}}{\frac{\alpha(\alpha - 1)}{2}}x^2 + \frac{\alpha(\alpha - 1)(\alpha - 2)}{6}x^3 + \cdots\cdots + \binom{\alpha}{n}x^n + \cdots\cdots$

3') $\frac{1}{1 + x} = 1 - x + x^2 - x^3 + x^4 + \cdots\cdots + (-1)^n x^n + \cdots\cdots$

3'') $\frac{1}{1 - x} = 1 + x + x^2 + x^3 + x^4 + \cdots\cdots + x^n + \cdots\cdots$

3''') $\sqrt{1 + x} = 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{1}{16}x^3 - \frac{5}{128}x^4 + \cdots\cdots + \frac{(-1)^{n+1}(2n - 3)!!}{(2n)!!}x^n + \cdots\cdots$

3'v) $\sqrt{1 - x} = 1 - \frac{1}{2}x - \frac{1}{8}x^2 - \frac{1}{16}x^3 - \frac{5}{128}x^4 - \cdots\cdots - \frac{(2n - 3)!!}{(2n)!!}x^n - \cdots\cdots$

3v) $\frac{1}{\sqrt{1 + x}} = 1 - \frac{1}{2}x + \frac{3}{8}x^2 - \frac{5}{16}x^3 + \frac{35}{128}x^4 + \cdots\cdots + (-1)^n\frac{(2n - 1)!!}{(2n)!!}x^n + \cdots\cdots$

3v') $\frac{1}{\sqrt{1 - x}} = 1 + \frac{1}{2}x + \frac{3}{8}x^2 + \frac{5}{16}x^3 + \frac{35}{128}x^4 + \cdots\cdots + \frac{(2n - 1)!!}{(2n)!!}x^n + \cdots\cdots$

3v'') $\dfrac{1}{(1-x)^2} = 1 + 2x + 3x^2 + 4x^3 + 5x^4 + \cdots\cdots + (n+1)x^n + \cdots\cdots$

3v''') $\dfrac{1}{(1+x)^2} = 1 - 2x + 3x^2 - 4x^3 + 5x^4 + \cdots\cdots + (-1)^n(n+1)x^n + \cdots\cdots$

4) $\text{sen}x = x - \frac{1}{6}x^3 + \frac{1}{120}x^5 - \frac{1}{5040}x^7 + \cdots\cdots + \frac{(-1)^n}{(2n+1)!}x^{2n+1} + \cdots\cdots$

5) $\cos x = 1 - \frac{1}{2}x^2 + \frac{1}{24}x^4 - \frac{1}{720}x^6 + \cdots\cdots + \frac{(-1)^n}{(2n)!}x^{2n} + \cdots\cdots$

6) $\text{tg}x = x + \frac{1}{3}x^3 + \frac{2}{15}x^5 + \frac{17}{315}x^7 + O(x^9)$

7) $\text{senh}x = x + \frac{1}{6}x^3 + \frac{1}{120}x^5 + \frac{1}{5040}x^7 + \cdots\cdots + \frac{1}{(2n+1)!}x^{2n+1} + \cdots\cdots$

8) $\cosh x = 1 + \frac{1}{2}x^2 + \frac{1}{24}x^4 + \frac{1}{720}x^6 + \cdots\cdots + \frac{1}{(2n)!}x^{2n} + \cdots\cdots$

9) $\text{tgh}x = x - \frac{1}{3}x^3 + \frac{2}{15}x^5 - \frac{17}{315}x^7 + O(x^9)$

10) $\text{arcsen}x = x + \frac{1}{6}x^3 + \frac{3}{40}x^5 + \frac{5}{112}x^7 + \cdots\cdots + \frac{(2n-1)!!}{(2n+1)(2n)!!}x^{2n+1} + \cdots\cdots$

11) $\text{arctg}x = x - \frac{1}{3}x^3 + \frac{1}{5}x^5 - \frac{1}{7}x^7 + \cdots\cdots + \frac{(-1)^n}{2n+1}x^{2n+1} + \cdots\cdots$

12) $\text{arcsenh}x = x - \frac{1}{6}x^3 + \frac{3}{40}x^5 - \frac{5}{112}x^7 + \cdots\cdots + \frac{(-1)^n(2n-1)!!}{(2n+1)(2n)!!}x^{2n+1} + \cdots\cdots$

13) $\text{arctgh}x = x + \frac{1}{3}x^3 + \frac{1}{5}x^5 + \frac{1}{7}x^7 + \cdots\cdots + \frac{1}{2n+1}x^{2n+1} + \cdots\cdots$

Esercizio:

Calcolare:

$$\lim_{x\to 0} \frac{tg x - \log(e^x \cos x)}{x \log(1+\sin x)} = ?$$

$$tg x = x + \frac{1}{3}x^3 + o(x^3)$$
$$e^x = 1 + x + x^2/2 + o(x^2)$$
$$\cos x = 1 - \frac{x^2}{2} + o(x^2)$$

$$e^x \cos x = 1 + x + \cancel{\frac{x^2}{2}} - \cancel{\frac{x^2}{2}} + o(x^2)$$

$$\log(e^x \cos x) = \log(1 + x + o(x^2))$$

$$\log(1+t) = t - \frac{t^2}{2} + o(t^2)$$

$$\Downarrow$$

$$\log(e^x \cos x) = x + o(x^2) - \frac{(x + o(x^2))^2}{2} + o(x^2) =$$

$$= x - \frac{x^2}{2} + o(x^2)$$

$$tg x = x + \frac{1}{3}x^3 + o(x^3) = x + o(x^2)$$

$$tg x - \log(e^x \cos x) = \cancel{x} - \cancel{x} + \frac{x^2}{2} + o(x^2)$$

$$\sin x = x - \frac{1}{6}x^3 + o(x^3) = x + o(x)$$

$$\log(1 + \sin x) = \log(1 + x + o(x)) = x + o(x)$$

$$x \cdot \log(1 + \sin x) = x^2 + o(x^2)$$

$$\frac{tg x - \log(e^x \cos x)}{x \cdot \log(1 + \sin x)} = \frac{\frac{x^2}{2} + o(x^2)}{x^2 + o(x^2)}$$

$$\Downarrow$$

$$\lim_{x\to 0} \frac{tg x - \log(e^x \cos x)}{x \cdot \log(1+\sin x)} = \lim_{x \to 0} \frac{\cancel{x^2}}{2\cancel{x^2}} = \frac{1}{2}$$